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JASA LETTER TO THE EDITOR

Numerical solution of Rothenberg's equation of the glottal airflow rate as a function of the glottal area using backward differentiation (L)

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ABSTRACT:
This letter shows the application of backward differentiation formulas to solve a differential equation by Rothenberg [(1981). Department for Speech, Music and Hearing Quarterly Progress and Status Report (KTH Royal Institute of Technology, Stockholm, Sweden), Vol. 22], which models the glottal airflow rate vs the glottal area. The formulas avoid a singularity of the equation that occurs when the glottal area is zero and that prevents the application of Runge–Kutta and other numerical methods. They can also be used when the equation is augmented with a glottal air viscosity term to eliminate non-differentiability at glottal opening and closure.
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I. INTRODUCTION

When the fundamental frequency of the vocal fold oscillation is below the first formant frequency of the vocal tract, the relation between of the glottal airflow rate u_g and the glottal area a_g may be characterized by a model devised by Rothenberg (1981), of the form

$$P_L = Ru_g + I\frac{du_g}{dt} + \frac{k_t\rho}{2}\frac{u_g|u_g|}{a_g^2}, \qquad (1)$$

where P_L is the lung pressure, R and I are the lumped input resistance and inertance of the vocal tract, respectively (combining both the subglottal and supraglottal airways), k_t is an empirical transglottal coefficient, and ρ is the air density (see also Titze and Alipour, 2006). The model has been useful to investigate the acoustical coupling between the glottal source and the vocal tract and also in formant synthesis of voice driven by glottal area (Rruqja *et al.*, 2014).

Equation (1) is a first-order differential equation of the Riccati type, which has been well studied in the literature (Murphy, 1960). However, its resolution is complicated by a singularity when the glottal area is zero. Rewriting Eq. (1) in standard explicit form $du_g/dt = f(t, u_g)$, where

$$f(t,u_g) = \frac{1}{I}\left(P_L - Ru_g - \frac{k_t\rho}{2}\frac{u_g|u_g|}{a_g^2}\right), \qquad (2)$$

then we have that $f(t,u_g)$ is not defined for values of t at which $a_g(t) = 0$. Therefore, if the model is used to compute the airflow rate based on glottal area waveforms that include periods of glottal closure, then numerical methods that require computation of $f(t,u_g)$ cannot be used. Such is the case of explicit algorithms, such as the Runge–Kutta methods, and also some implicit algorithms, such as the trapezoidal rule and higher-order Adams–Moulton methods. One strategy to avoid the singularity has been to impose a minimum small positive value for the glottal area (e.g., Titze, 2006) at the expense of generating an airflow leakage during the glottal closure. Also, depending on the value of such a minimum, the division by a small number may produce numerical instability. Other studies have avoided the division by a_g by approximating the solution as a resistive flow plus a small corrective component (e.g., Bennane *et al.*, 2015). However, such approximation has a general low accuracy.

Then how may Eq. (1) be solved efficiently, accurately, and including periods of full glottal closure? An answer is through the application of multistep backward differentiation formulas (BDFs) (Quarteroni *et al.*, 2000), as this letter will show.

Incidentally, let us note that the absolute value at the right side of Eq. (1) is often simplified by replacing $u_g|u_g|$ with u_g^2, under the assumption that the transglottal pressure is non-negative and consequently $u_g \geq 0$ (Titze and Alipour, 2006). However, brief periods of negative flow may appear in voice simulations if the lung pressure is reduced to a small value while u_g is rapidly increasing. For example, letting $Idu_g/dt = C > P_L \geq 0$, $a_g > 0$, and solving for u_g, we obtain

$$u_g = \frac{Ra_g^2}{k_t\rho} - \sqrt{\frac{R^2a_g^4}{k_t^2\rho^2} + \frac{2a_g^2(C-P_L)}{k_t\rho}} < 0. \qquad (3)$$

Thus, this letter considers the model in the general form expressed by Eq. (1).

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II. SOLUTION OF ROTHENBERG'S EQUATION

BDFs have the general form

$$u_{g,k} = \sum_{i=1}^{m} \alpha_i u_{g,k-i} + h\beta f(t_k, u_{g,k}), \tag{4}$$

where $u_{g,k} = u_g(t_k)$ for $k = 1, 2, 3, \ldots$, α_i and $\beta \neq 0$ are rational coefficients, $h = t_k - t_{k-1} > 0$ is the time step, and $1 \leq m \leq 6$ is the order of the formula (the formulas for $m > 6$ are not convergent). Equation (4) contains an implicit relation for the unknown $u_{g,k}$, which appears at the left side and also as an argument to function f at the right side. In general applications of BDFs, the relation is nonlinear and must be solved iteratively. However, an explicit solution can be obtained algebraically in the present case, as follows.

Inserting Eq. (2) into Eq. (4) results in the quadratic equation

$$u_{g,k}|u_{g,k}| + 2bu_{g,k} - c = 0, \tag{5}$$

where

$$b = \frac{a_{g,k}^2}{k_t \rho}\left(R + \frac{I}{h\beta}\right), \tag{6}$$

$$c = \frac{2a_{g,k}^2}{k_t \rho}\left(P_L + \frac{I}{h\beta}\sum_{i=1}^{m} \alpha_i u_{g,k-i}\right), \tag{7}$$

and $a_{g,k} = a_g(t_k)$. Considering both cases of $u_{g,k} \geq 0$ and $u_{g,k} < 0$ and solving Eq. (5) for $u_{g,k}$ yields[1]

$$u_{g,k} = \mathrm{sgn}(c)\left(-b + \sqrt{b^2 + |c|}\right), \tag{8}$$

where sgn denotes the sign function

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases} \tag{9}$$

The above Eqs. (6)–(8) do not have a division by $a_{g,k}$, and they produce the correct value of $u_{g,k} = 0$ for $a_{g,k} = 0$ (i.e., there is no airflow when the glottis is closed). Further, the equations apply even if $I = 0$. In that case, Eq. (1) is algebraic, and Eq. (8) gives its exact solution.

Figure 1 shows a numerical example. A symmetrical glottal area curve was simulated with a truncated sinusoidal waveform of an amplitude of 0.15 cm^2, fundamental frequency of 125 Hz, and open quotient of 0.6. The airflow rate was computed with the above algorithm for a third-order BDF, with $m = 3$, $\beta = 6/11$, $\alpha_1 = 18/11$, $\alpha_2 = -9/11$, $\alpha_3 = 2/11$ (Quarteroni *et al.*, 2000), and time step $h = 0.02$ ms. The vocal tract parameters were set to $R = 6$ g/(s cm^4), $I = 0.01$ g/cm^4, $P_L = 500$ Pa, $k_t = 1$, and $\rho = 0.001\,14$ g/cm^3 (Titze and Alipour, 2006). The airflow rate follows the area variation with a skewness to the left (i.e., the pulse leans to the right) due to the vocal tract inertance, and the algorithm is able to simulate the closed glottis period with a zero airflow.

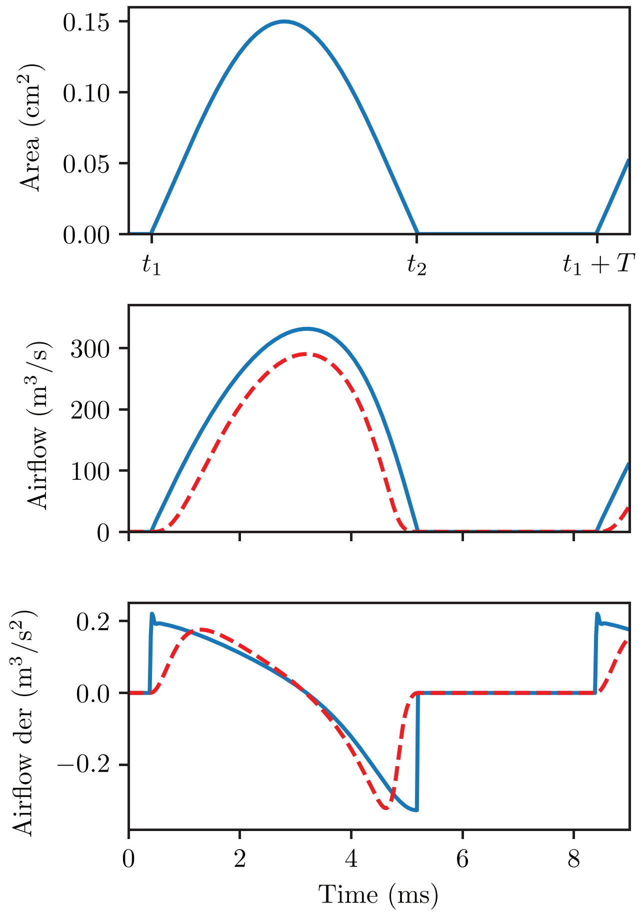

FIG. 1. (Color online) Numerical example of a glottal area curve (top), glottal airflow rate computed with a third-order BDF (middle), and airflow rate derivative (bottom). In the upper plot, t_1 and t_2 denote the time instants of glottal opening and closure, respectively, and T is the oscillation period. In the middle and bottom plots, the blue solid curves show solutions to the model in Eq. (1), and the red dashed curves show solutions after adding an air viscosity term as in Eq. (10).

III. CORRECTION FOR SMOOTHNESS

As shown by the example in Fig. 1, at the time instants of glottal opening and glottal closure (t_1 and t_2, respectively), the resultant airflow rate is continuous but non-differentiable. The non-differentiability is revealed by jump discontinuities in the derivative, and it is a consequence of a non-differentiable glottal area waveform at t_1 and t_2 (Lucero and Schoentgen, 2015).

The glottal airflow derivative is an important component in voice production studies, and its correct characterization demands its continuity with a smooth transition between the periods of open and closed glottis (Fant *et al.*, 1985). A simple solution to obtain a differentiable airflow rate is to add a pressure loss term for glottal air viscosity of the form $\gamma u_g/a_g^3$ to Eq. (1),

$$P_L = Ru_g + I\frac{du_g}{dt} + \frac{k_t\rho}{2}\frac{u_g|u_g|}{a_g^2} + \gamma\frac{u_g}{a_g^3}, \tag{10}$$

where γ is a positive coefficient. The effect of such a term on the flow rate was observed by Ananthapadmanabha and

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Fant (1982) and was also considered by Lucero and Schoentgen (2015) in the context of an airflow model by Titze (1984).

Again, the numerical solution to Eq. (10) must be implemented so as to avoid any division by a_g. BDFs still apply, which result in the quadratic equation

$$a_{g,k}u_{g,k}|u_{g,k}| + 2b' u_{g,k} - a_{g,k}c = 0, \tag{11}$$

where

$$b' = \frac{1}{k_t\rho}\left(Ra_{g,k}^3 + \frac{I}{h\beta}a_{g,k}^3 + \gamma\right). \tag{12}$$

Solving Eq. (11) for both cases of $u_{g,k} \geq 0$ and $u_{g,k} < 0$ yields

$$u_{g,k} = \text{sgn}(c)\left(\frac{-b' + \sqrt{b'^2 + a_{g,k}^2|c|}}{a_{g,k}}\right). \tag{13}$$

Finally, multiplying and dividing by $b' + \sqrt{b'^2 + a_{g,k}^2|c|}$, we obtain[2]

$$u_{g,k} = \frac{a_{g,k}c}{b' + \sqrt{b'^2 + a_{g,k}^2|c|}}. \tag{14}$$

A quick proof of the differentiability of $u_g(t)$ at $a_g = 0$ may be stated as follows. Multiply both sides of Eq. (10) by a_g^3 and consider an approximate solution of the form $u_g(t) = ra_g^3(t)$ for $a_g(t)$ small enough, where r is a non-negative constant. Then

$$\frac{du_g}{dt} = 3ra_g^2\frac{da_g}{dt}, \tag{15}$$

and replacing in Eq. (10) yields

$$P_L = rRa_g^3 + 3ra_g^2 I\frac{da_g}{dt} + \frac{k_t\rho r^2}{2}a_g^4 + \gamma r. \tag{16}$$

For $a_g \to 0$ and assuming that da_g/dt is bounded, we obtain $r = P_L/\gamma \geq 0$ and constant. Therefore, the proposed approximation holds. The airflow derivative du_g/dt is then approximated by Eq. (15), which is defined and continuous for $a_g = 0$.

A numerical example is shown in Fig. 1, with $\gamma = 0.01$ g cm^2/s and other parameters as in Sec. II for comparison.[3] The resultant airflow rate is smooth and differentiable at the instants of glottal opening and closure, without any other relevant effect on the shape of the airflow pulse except for an amplitude reduction.

IV. SUMMARY

BDFs provide a convenient method for the numerical solution of Rothenberg's glottal airflow model in Eq. (1), with accuracy up to order 6. The resultant algorithm, expressed by Eq. (8), avoids the singularity that appears when the glottal area is zero. If the model is augmented with a glottal air viscosity term for differentiability at glottal opening and closure, then Eq. (14) must be used instead.

[1]See Eqs. (12)–(15) in Lucero and Schoentgen (2015), with $c_0 = c$, $c_1 = 2b$, and $c_2 = 1$.

[2]See Eqs. (17) and (18) in Lucero and Schoentgen (2015), with $c_0 = a_g c$, $c_1 = 2b'$, and $c_2 = a_g$.

[3]The value of $\gamma = 0.01$ g cm^2/s is much higher than the value obtained by using the Poiseuille's equation for flow between parallel plates (van den Berg *et al.*, 1957). That equation produces $\gamma = 12\mu L^2 T$, where μ is the air viscosity coefficient, L is the glottal length in the anterior-posterior direction, and T is the glottal depth in the direction of the airflow (Lucero and Schoentgen, 2015). Letting $\mu = 0.000186$ g/(cm s), $L = 1$ cm, and $T = 0.3$ cm (Titze and Alipour, 2006), we obtain $\gamma = 0.000\,67$ g cm^2/s. The higher value of γ in Fig. 1 was chosen to produce a more visible effect.

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